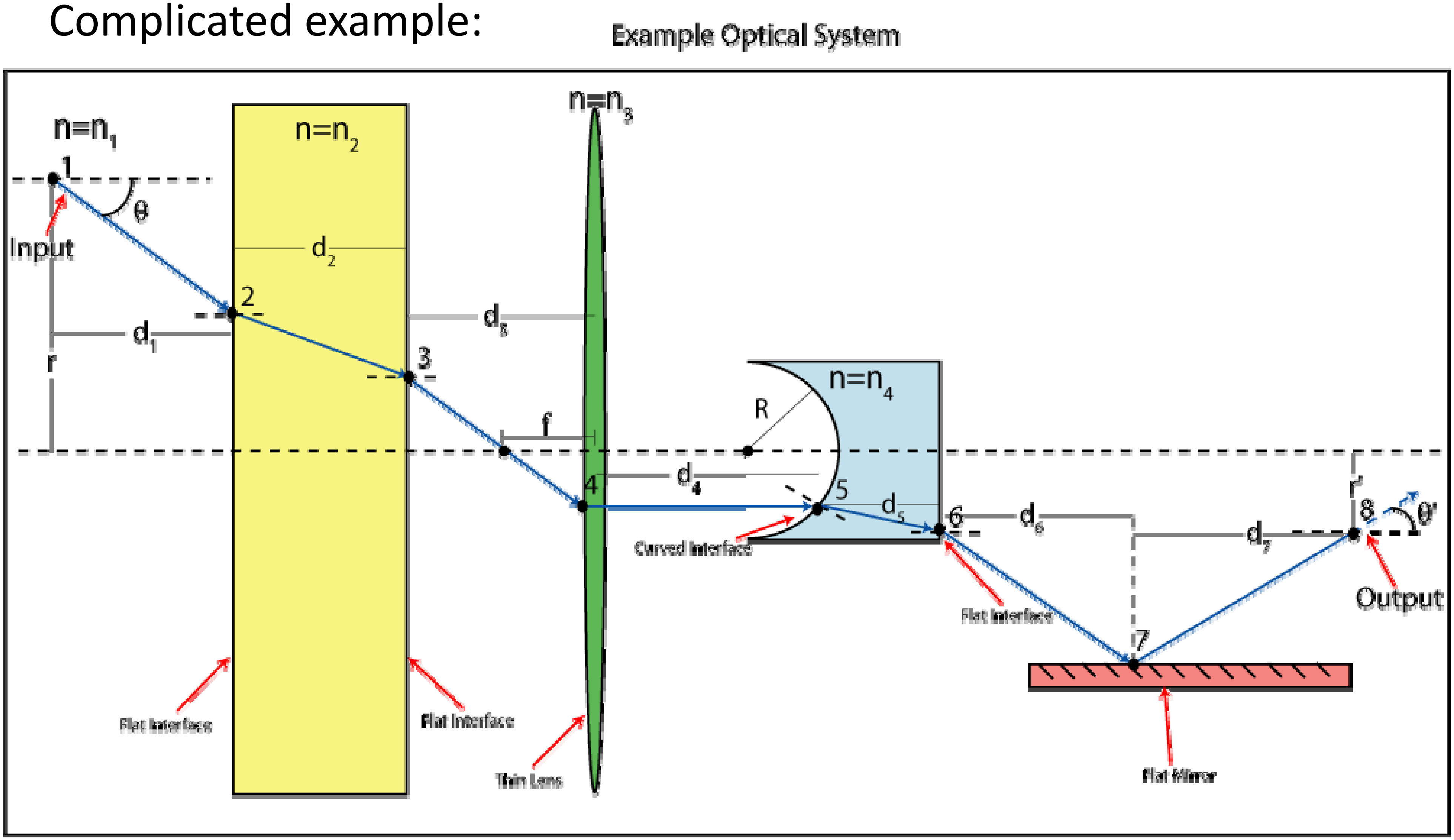
Complicated example:



1) Medium between points 1 and 2 (with constant index of refraction n_1)
2) Flat interface at point 2
3) Medium between points 2 and 3 (with constant index of refraction n_2)
4) Flat interface at point 3
5) Medium between points 3 and 4 (with constant index of refraction n_1)
6) Thin lens with focal length f and index of refraction n_3
7) Medium between points 4 and 5 (with constant index of refraction n_1)
8) Curved interface at point 5
9) Medium between points 5 and 6 (with constant index of refraction n_4)
10) Flat interface at point 6
11) Medium between points 6 and 7 (with constant index of refraction n_1)
12) Flat mirror at point 7
13) Medium between points 7 and 8 (with constant index of refraction n_1)

Let M = The 2 by 2 transfer matrix of the entire system

$$M=\overset{(13)}{\begin{pmatrix}1 & d_7\\ 0 & 1\end{pmatrix}}\overset{(12)}{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}\overset{(11)}{\begin{pmatrix}1 & d_6\\ 0 & 1\end{pmatrix}}\overset{(10)}{\begin{pmatrix}1 & 0\\ 0 & \frac{n_4}{n_1}\end{pmatrix}}\overset{(9)}{\begin{pmatrix}1 & d_5\\ 0 & 1\end{pmatrix}}\overset{(8)}{\begin{pmatrix}1 & 0\\ \frac{n_1-n_4}{R\cdot n_4} & \frac{n_1}{n_4}\end{pmatrix}}\overset{(7)}{\begin{pmatrix}1 & d_4\\ 0 & 1\end{pmatrix}}\overset{(6)}{\begin{pmatrix}1 & 0\\ \frac{-1}{f} & 1\end{pmatrix}}\overset{(5)}{\begin{pmatrix}1 & d_3\\ 0 & 1\end{pmatrix}}\overset{(4)}{\begin{pmatrix}1 & 0\\ 0 & \frac{n_2}{n_1}\end{pmatrix}}\overset{(3)}{\begin{pmatrix}1 & d_2\\ 0 & 1\end{pmatrix}}\overset{(2)}{\begin{pmatrix}1 & 0\\ 0 & \frac{n_1}{n_2}\end{pmatrix}}\overset{(1)}{\begin{pmatrix}1 & d_1\\ 0 & 1\end{pmatrix}}$$

$$\begin{pmatrix}r'\\ \theta'\end{pmatrix}=M\begin{pmatrix}r\\ \theta\end{pmatrix}$$

Back to the paraxial wave equation

The paraxial wave equation (section 7.4 Milonni)

$$\nabla^2 E(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(\mathbf{r},t) = 0$$

$$E(\mathbf{r},t) = \mathcal{E}(\mathbf{r})e^{-i\omega t}$$

$$\nabla^2 \mathcal{E}(\mathbf{r}) + k^2\mathcal{E}(\mathbf{r}) = 0.$$

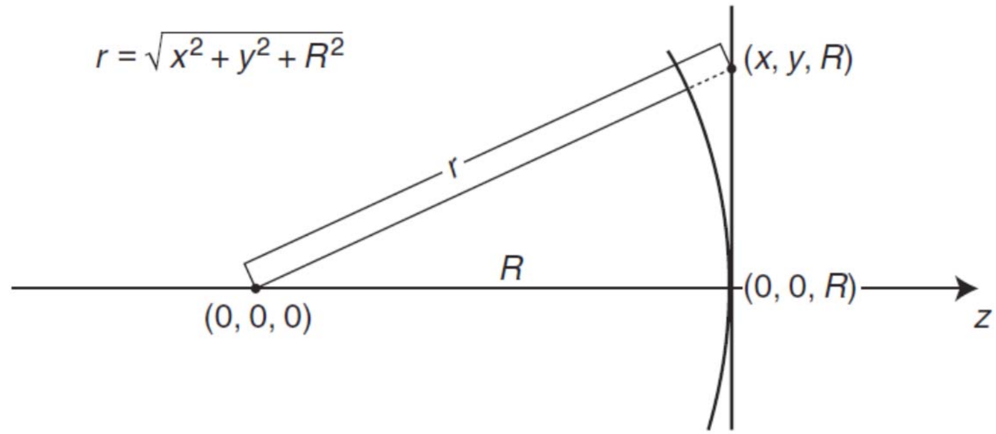
$$k^2 = \frac{\omega^2}{c^2}$$

In the previous class we have already discussed a solution (**plane wave**):

$$\mathcal{E}(\mathbf{r}) = \mathcal{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$$

The paraxial wave equation has another solution: spherical wave.

$$\mathcal{E}(\mathbf{r}) = \frac{A}{r} e^{ikr}$$



With z=R,

$$r = (x^2 + y^2 + R^2)^{1/2} = R\left(1 + \frac{x^2 + y^2}{R^2}\right)^{1/2}.$$

$$\left(1 + \frac{x^2 + y^2}{R^2}\right)^{1/2} = 1 + \frac{x^2 + y^2}{2R^2} + \cdots$$

$$kr \approx kR + \frac{k(x^2 + y^2)}{2R}$$

$$\mathcal{E}(\mathbf{r}) = \frac{A}{R} e^{ikR} e^{ik(x^2+y^2)/2R}$$

An accurate approximation to the **spherical wave**.

We will now look for solutions therefore of the wave equation that look more like beams.

$$\mathcal{E}(\mathbf{r}) = \mathcal{E}_0(\mathbf{r})e^{ikz}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathcal{E}_0(\mathbf{r})e^{ikz} + k^2\mathcal{E}_0(\mathbf{r})e^{ikz} = 0$$

We will make the following assumptions:

$$\left|\frac{\partial \mathcal{E}_0}{\partial z}\right| \ll k|\mathcal{E}_0|, \qquad \left|\frac{\partial^2 \mathcal{E}_0}{\partial z^2}\right| \ll k\left|\frac{\partial \mathcal{E}_0}{\partial z}\right|$$

$$\frac{\partial^2}{\partial z^2}\mathcal{E}_0(\mathbf{r})e^{ikz} = \left(\frac{\partial^2\mathcal{E}_0}{\partial z^2} + 2ik\frac{\partial\mathcal{E}_0}{\partial z} - k^2\mathcal{E}_0\right)e^{ikz} \approx \left(2ik\frac{\partial\mathcal{E}_0}{\partial z} - k^2\mathcal{E}_0\right)e^{ikz}$$

This leads to the paraxial wave equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik\frac{\partial}{\partial z}\right)\mathcal{E}_0(\mathbf{r}) \approx 0$$

The following operator is called the Transverse Laplacian:

$$\nabla_T^2\mathcal{E}_0 + 2ik\frac{\partial\mathcal{E}_0}{\partial z} = 0,$$

$$\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Gaussian beams

As we discussed before a Gaussian beam intensity profile has the form:

$$I(x, y, z) \sim |\mathcal{E}_0|^2 e^{-2(x^2+y^2)/w^2}$$

we will try to construct a solution of the form

$$\mathcal{E}_0(\mathbf{r}) = A e^{ik(x^2+y^2)/2q(z)} e^{ip(z)}$$

Assuming the following form, we can recover the Gaussian profile:

$$\frac{1}{q} = \frac{2i}{kw^2} = \frac{i\lambda}{\pi w^2}$$

we can derive the following:

$$\frac{\partial \mathcal{E}_0}{\partial z} = iA\left[\frac{dp}{dz} - \frac{k}{2}(x^2 + y^2)\frac{1}{q^2}\frac{dq}{dz}\right] e^{ik(x^2+y^2)/2q(z)} e^{ip(z)}$$

$$\nabla_T^2 \mathcal{E}_0 = A\left[\frac{2ik}{q} - \frac{k^2}{q^2}(x^2 + y^2)\right] e^{ik(x^2+y^2)/2q(z)} e^{ip(z)}$$

(Homework problem)

$$\nabla_T^2 \mathcal{E}_0 + 2ik\frac{\partial \mathcal{E}_0}{\partial z} = A\left[\frac{k^2}{q^2}(x^2 + y^2)\left(\frac{dq}{dz} - 1\right) - 2k\left(\frac{dp}{dz} - \frac{i}{q}\right)\right] e^{ik(x^2+y^2)/2q(z)} e^{ip(z)}$$

Conditions that should be satisfied:

$$\frac{dq}{dz} = 1 \qquad \frac{dp}{dz} = \frac{i}{q}$$

These are the solutions to the previous equations:

$$q(z) = q_0 + z. \qquad p(z) = i \ln \frac{q_0 + z}{q_0}$$

As q could be imaginary we choose the form:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{i\lambda}{\pi w^2(z)}$$

We can then re-write the solution:

$$\mathcal{E}_0(\mathbf{r}) = A e^{ik(x^2+y^2)/2q(z)} e^{ip(z)}$$

$$e^{ik(x^2+y^2)/2q(z)} = e^{ik(x^2+y^2)/2R(z)} e^{-(x^2+y^2)/w^2(z)}$$

$$e^{ip(z)} = \exp\left(-\ln\frac{q_0+z}{q_0}\right) = \frac{q_0}{q_0+z} = \frac{1}{1+z/q_0} = \frac{1}{1+z/R_0+i\lambda z/\pi w_0^2}$$

If we pick:

$$R_0 = \infty \qquad \frac{1}{q_0} = \frac{i\lambda}{\pi w_0^2}$$

$$\frac{1}{q(z)} = \frac{1}{q_0 + z} = \frac{1/q_0}{1 + z(1/q_0)} = \frac{i\lambda/\pi w_0^2}{1 + iz\lambda/\pi w_0^2} = \frac{i\lambda/\pi w_0^2 + (1/z)(\lambda z/\pi w_0^2)^2}{1 + (\lambda z/\pi w_0^2)^2}$$

$$= \frac{1}{R(z)} + \frac{i\lambda}{\pi w^2(z)}.$$

Equating separately the real and imaginary parts, we have

$$R(z) = z + \frac{z_0^2}{z} \qquad w(z) = w_0\sqrt{1 + z^2/z_0^2}$$

$$z_0 = \frac{\pi w_0^2}{\lambda}$$ **Rayleigh range**

Also, we can re-write:

$$e^{ip(z)} = \frac{1}{1 + iz/z_0} = \frac{1}{\sqrt{1 + z^2/z_0^2}} e^{-i\phi(z)}. \qquad \phi(z) = \tan^{-1}\left(\frac{z}{z_0}\right)$$

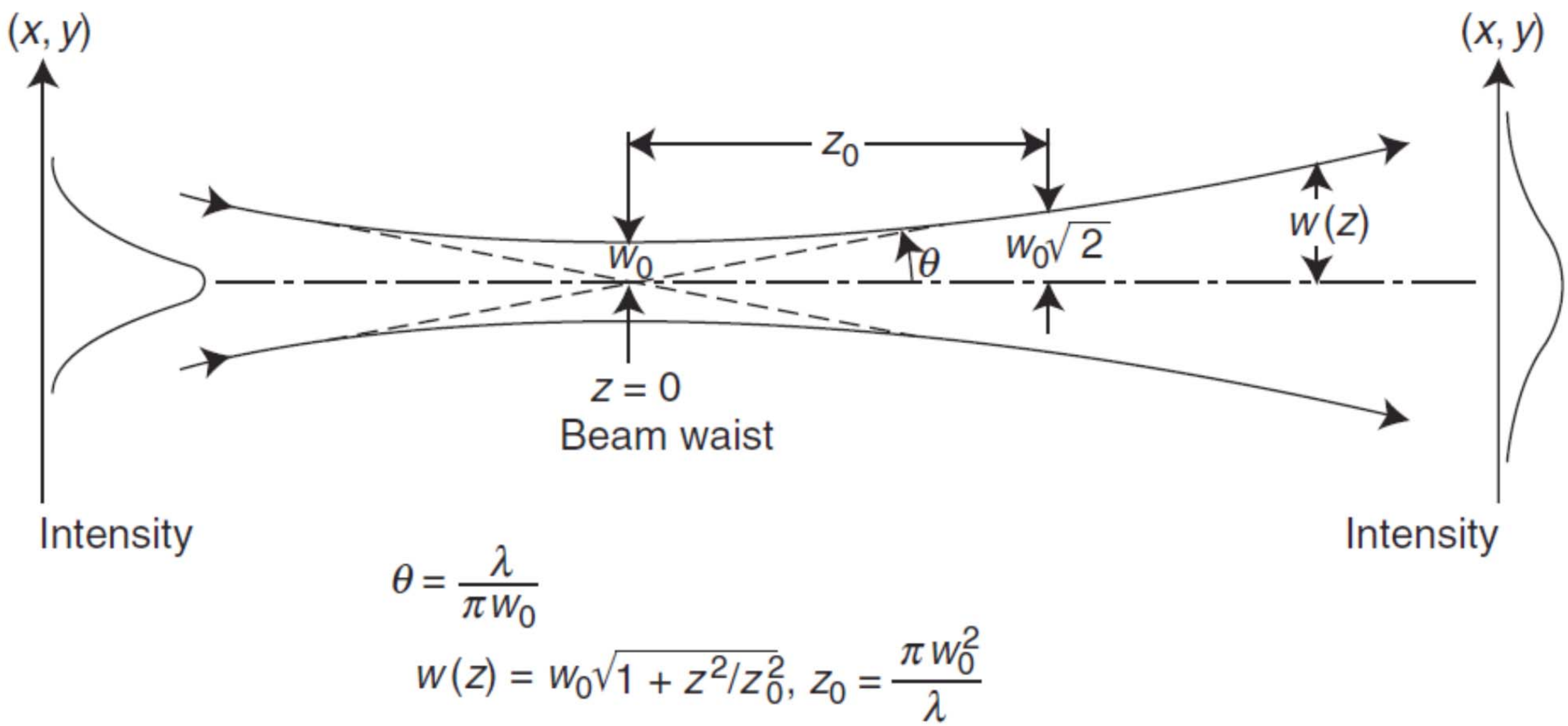
so now we have the full Gaussian solution to the wave equation:

$$\mathcal{E}(\mathbf{r}) = \frac{A e^{ikz} e^{-i\phi(z)}}{\sqrt{1 + z^2/z_0^2}} e^{ik(x^2+y^2)/2R(z)} e^{-(x^2+y^2)/w^2(z)}$$

A few useful results:

$$w(z_0) = w_0\sqrt{2}$$

$$\theta \approx \frac{w(z)}{z} \approx \frac{w_0}{z_0} = \frac{\lambda}{\pi w_0} \qquad (z \gg z_0)$$



More useful results for your laser lab introduction:

$$z \gg z_0$$

$$I(x, y, z) \approx \frac{I_0}{z^2} z_0^2 e^{-2(x^2+y^2)/w^2(z)} \qquad w(z) \approx \frac{w_0 z}{z_0} = \frac{\lambda z}{\pi w_0}$$

$$E(\mathbf{r}) = \mathcal{E}(\mathbf{r}) e^{-i\omega t} \quad \text{(electric field)}$$

$$\mathcal{E}(\mathbf{r}) = A \frac{w_0}{w(z)} e^{i[kz - \tan^{-1}(z/z_0)]} e^{ik(x^2+y^2)/2R(z)} e^{-(x^2+y^2)/w^2(z)}$$

$$I(\mathbf{r}) = \frac{c\epsilon_0}{2} |A|^2 e^{-2(x^2+y^2)/w^2(z)} \quad \text{(intensity)}$$

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_0^2}} \quad \text{(spot size)}$$

$$R(z) = z + \frac{z_0^2}{z} \quad \text{(radius of curvature)}$$

$$z_0 = \pi w_0^2/\lambda \quad \text{(Rayleigh range)}$$

$$\theta = \lambda/\pi w_0 \quad \text{(divergence angle)}$$

Given the wavelength and the beam waist we can determine the spot size and radius of curvature everywhere.