Homework #1

Problem 1. Given the paraxial wave equation:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik \frac{\partial}{\partial z} \right) E_0(\mathbf{r}) \approx 0
\]

And a proposed solution of the form:

\[
E_0(\mathbf{r}) = Ae^{ik(x^2+y^2)/2q(z)}e^{ip(z)}
\]

Prove the following:

\[
\frac{\partial E_0}{\partial z} = iA \left[ \frac{dp}{dz} - \frac{k}{2} (x^2 + y^2) \frac{1}{q^2} \frac{dq}{dz} \right] e^{ik(x^2+y^2)/2q(z)}e^{ip(z)}
\]

\[
\nabla_\tau^2 E_0 = A \left[ \frac{2ik}{q} - \frac{k^2}{q^2} (x^2 + y^2) \right] e^{ik(x^2+y^2)/2q(z)}e^{ip(z)}
\]

Problem 2. A Gaussian beam is transmitted through a thin lens of focal length f (see diagram below).

(a) Show that the locations of the waists of the incident and transmitted beams, z and z', are related by:

\[
\frac{z'}{f} - 1 = \frac{z/f - 1}{(z/f - 1)^2 + (z_0/f)^2}
\]
(b) The beam is collimated by making the location of the new waist $z'$ as distant as possible from the lens. This is achieved by using the smallest ratio $z_0/f$ (short depth of focus and long focal length). For a given ratio $z_0/f$, show that the optimal value of $z$ for collimation is $2 = f + z_0$.

(c) If $\lambda = 1 \mu m$, $z_0 = 1$ cm and $f = 50$ cm, determine the optimal value of $z$ for collimation, and the corresponding magnification $M$, distance $z'$, and width $W_0'$ of the collimated beam.

**Problem 3.** Demonstrate that if a ray passes a spherical dielectric interface, the ABCD matrix describing the propagation is given by:

$$
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
\frac{1}{n_2 - n_1} & 0 \\
\frac{n_1}{n_2 R} & \frac{n_1}{n_2}
\end{bmatrix}
$$

where $R > 0$ if the surface is concave and $R < 0$ if the surface is convex.

Additionally, solve problems 7.1, 7.9, 7.10 and 7.14 from the Milonni book (2nd edition).